

# A SIMPLE ACTION FOR A FREE ANYON

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## Abstract

By studying classical realizations of the  $\mathfrak{sl}(2, \mathbb{R})$  algebra in a two dimensional phase space  $(q, \pi)$ , we have derived a continuous family of new actions for free anyons in 2+1 dimensions. For the case of light-like spin vector ( $S_\mu S^\mu = 0$ ), the action is remarkably simple. We show the appearance of the Zitterbewegung in the solutions of the equations of motion, and relate the actions to others in the literature at classical level.

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# 1 Introduction

One of the main motivations in the physics of particles with fractional statistics in  $2 + 1$  dimensions (anyons [1]) is to address the questions of fractional quantum hall effect and high temperature superconductors [2,3] which are presumably planar effects. Typically one makes use of a statistical field  $A_\mu$  of the Chern-Simons type, coupled to some matter field whose statistics will be changed by  $A_\mu$ . One interesting question about this coupling is whether some interaction is left for the anyon besides the change of statistics. This question suggests the search for actions for free anyons as elementary relativistic particles. In particular, from the point of view of point particles, there are already several works in the literature suggesting different actions for anyons [4-12]. Care should be taken with the fact that in  $2 + 1$  dimensions the irreps of the Lorentz group are finite only for integer or half-integer spins. Thus, for fractional spins we have to deal with infinite dimensional representations.

There are basically two kinds of point-particle actions for anyons in the literature. Since anyons have the same number of degrees of freedom as massive spinless particles, they can be described by the same space-time variables  $(x_\mu, p_\nu)$ , see [4, 6] and also [9]. The price paid in this minimal formulation is the appearance of non canonical brackets among the space-time variables ( $\{x_\mu, x_\nu\} \neq 0$ ) and anomalous spin algebra ( $\{S_\mu, S_\nu\} = 0$  instead of  $\{S_\mu, S_\nu\} = \epsilon_{\mu\nu\alpha} S^\alpha$ ) besides the lack of explicit Lorentz covariance. On the other hand, by adding extra variables, which must be subtracted afterwards, one can surmount these difficulties and even write explicitly Lorentz covariant actions, see e.g. [8], [10].<sup>1</sup> However, in these cases one has to face rather complicated Lagrangians. In the literature there are also extended formulations which are not explicitly covariant, e.g., [5], [12].

In this work we derive the minimal extension of the space-time phase space such that the canonical structure of the space-time and the canonical spin algebra are preserved. In order to circumvent the problem of lack of translation invariance raised up in [11] we use  $\dot{p}_\mu = 0$  replacing  $\ddot{x}_\mu = 0$  as a definition of a free particle which permits the appearance of the Zitterbewegung that will indeed show up. We end up with a family of rather interesting actions for free anyons which is specially simple when the spin is light-like ( $S^\mu S_\mu = 0$ ).

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<sup>1</sup>We remark that in [10] there is one exceeding degree of freedom

In section 2 we briefly review the covariant formulation of [8] showing the existence of a gauge [8] showing the existence of a gauge condition where the canonical structure of the space-time and the spin variables are preserved. In section 3 we study classical realizations of the  $\mathfrak{sl}(2, \mathfrak{R})$  spin algebra in a two dimensional phase-space deducing the family of Lagrangians cited before and showing the presence of the Zitterbewegung in the classical solutions. Our Lagrangian, for the case  $S^\mu S_\mu = 0$  is directly related to the covariant approach of [8] through a convenient gauge fixing followed by a Darboux transformation as shown in section 4. Section 5 contains some final comments and perspectives.

## 2 Covariant Description of Anyons

The Poincaré algebra in 2+1 dimensions is given by

$$\begin{aligned} [J_\mu, J_\nu] &= i\epsilon_{\mu\nu\alpha} J^\alpha \\ [J_\mu, P_\nu] &= i\epsilon_{\mu\nu\alpha} P^\alpha \\ [P_\mu, P_\nu] &= 0 \end{aligned} \tag{1}$$

where  $J_\mu$  are the dual components of the total angular momentum  $J_\mu = \epsilon_{\mu\nu\alpha} J^{\nu\alpha}$  and we use  $\eta_{\mu\nu} = (+, -, -)$ ,  $\epsilon_{012} = +1$ . Classically the Poincaré algebra can be realized in terms of Poisson brackets with the identifications:

$$J_\mu = \epsilon_{\mu\nu\alpha} x^\nu p^\alpha + S_\mu(q_i, \pi_j) \quad , \quad P_\mu = p_\mu \quad , \tag{2}$$

where we have introduced an extended phase space which besides the usual canonical variables  $(x_\mu, p_\nu)$ , necessary to describe a spinless particle, possesses also extra canonical variables  $(q_i, \pi_j)$ ;  $i = 1, \dots, N$ , used here to describe the dual spin components  $S_\mu$ . The only non vanishing brackets are  $\{q_i, \pi_j\} = \delta_i^j$ ;  $\{x_\mu, p_\nu\} = \delta_\mu^\nu$ . Moreover  $S_\mu$  satisfy the canonical spin algebra ( $\mathfrak{sl}(2, \mathfrak{R})$ ):

$$\{S_\mu, S_\nu\} = \epsilon_{\mu\nu\alpha} S^\alpha. \tag{3}$$

The Poincaré algebra (1) contains two quadratic Casimir invariants  $P^2 = p_\mu p^\mu$  and  $J \cdot P = S_\mu p^\mu$  which specify the mass and the helicity of the anyon

respectively. In order to assure that the physical states of the anyon belong to an irreducible representation of Poincarè algebra with given mass ( $m$ ) and helicity ( $\alpha$ ) we associate the Casimir invariants with two first class constraints following [6]:

$$\phi_1 = p^2 - m^2 \approx 0 \quad , \quad \phi_2 = S \cdot p + \alpha m \approx 0$$

where  $\alpha$  is a given real constant.

Another important ingredient when constructing a relativistic model for an anyon, is the correct counting of degrees of freedom. As shown in [6][13] the anyon posses, irrespective of  $\alpha$ , just one polarization state like a spinless particle. Therefore the extra variables  $(q_i, \pi_j)$  must be subtracted by means of suitable additional constraints. Since  $\phi_2$  is first class, the extra constraints  $\Phi_A$ ,  $A = 1, 2, \dots, l$  must be such that we get rid of all the remaining  $2N - 2$  variables among  $(q_i, \pi_j)$ . The number ' $l$ ' of such constraints depend upon their class, i. e., first or second class.

Considering all those ingredients and bearing in mind that first quantized theories of relativistic particles have vanishing canonical Hamiltonian, the general form of an anyon total Hamiltonian is given by [11]:

$$H = \frac{e}{2} \phi_1 + \sigma \phi_2 + \lambda^A \Phi_A \quad (4)$$

where  $e$ ,  $\sigma$  and  $\lambda^A$  are non-physical arbitrary functions of time. There are many choices for the extra variables  $(q_i, \pi_j)$ , and also different possibilities for the constraints  $\Phi_A$  which will lead in general to different parameterizations of the free anyon action, which should be physically equivalent. Probably the minimal way to describe the vector  $S_\mu$  in an explicitly covariant form is to introduce a vector  $n_\mu$  and its conjugate momentum  $\pi_\mu$ , with the only non vanishing brackets  $\{n_\mu, \pi_\nu\} = \eta_{\mu\nu}$ .

Now we can write  $S_\mu = \epsilon_{\mu\nu\alpha} n^\nu \pi^\alpha$ , which satisfy the  $\mathfrak{sl}(2, \mathfrak{R})$  algebra (3). In the rest of this section we will concentrate on this minimal covariant extension. Next step is to choose the constraints  $\Phi^A$  in order to eliminate four of the extra variables  $(n_\mu, \pi_\nu)$  since two of them will be already eliminated by  $\phi_2$ . It is clear that, the less constraints we have in (4) the simpler will be the Lagrangian obtained by the inverse Legendre transformation of  $H$ . Thus, the simplest choice for  $\Phi^A$  corresponds to two first class constraints  $\Phi_1, \Phi_2$ . The requirement of Lorentz covariance and the assumption of linear independence of the constraints demands that  $\Phi_i$  be scalars. The scalars

should not involve  $x^\mu$  since  $\dot{p}_\mu = 0$  for a free anyon. Therefore  $\Phi_i$  must be functions of  $\pi^2, n^2, \pi \cdot n, p \cdot n$ , and  $\pi \cdot p$ . We have verified that the choice  $\Phi_1 = \pi \cdot n \approx 0$ ,  $\Phi_2 = p \cdot n \approx 0$  leads to the simplest Lagrangian, which is given by

$$L = -m \frac{\epsilon_{\alpha\beta\gamma} \dot{x}^\alpha n^\beta \dot{n}^\gamma}{\left[\dot{n}^2 n^2 - (n \cdot \dot{n})^2\right]^{\frac{1}{2}}} + \frac{\alpha}{n^2} \left[\dot{n}^2 n^2 - (n \cdot \dot{n})^2\right]^{\frac{1}{2}}. \quad (5)$$

From this Lagrangian we get four primary first class constraints  $\phi_1, \phi_2, \Phi_1, \Phi_2$  and no further ones appear, such that the Hamiltonian is of the form (4):

$$H = \frac{e}{2} (p^2 - m^2) + \sigma (S \cdot p + \alpha m) + \lambda_1 \pi \cdot n + \lambda_2 p \cdot n. \quad (6)$$

with  $S_\mu = \epsilon_{\mu\nu\alpha} n^\nu \pi^\alpha$ . According to our knowledge this theory was first suggested in [8] where it was quantized in a gauge independent way. However to get closer to the physical degrees of freedom, it is important to further fix the gauge of the non universal constraints  $\Phi_1, \Phi_2$  while keeping the essential constraints  $\phi_1, \phi_2$  first class, in analogy to the pseudo-classical description of relativistic spinning particles of [14][15], where the local 1D-supersymmetry constraint plays the rôle of the  $\phi_2$  constraint giving rise to the Dirac propagator of spin  $\frac{1}{2}$  particles.

After introducing the gauge conditions  $\chi_i = 0$ ,  $i = 1, 2$  the constraints  $\Phi_i$  will turn into second class. In order that  $p^\mu$  be still interpreted as translation generators, i. e.,  $\{x_\mu, p_\nu\}_{DB} = \eta_{\mu\nu}$  the gauge conditions must be  $x^\mu$  independent. Moreover, Poincarè covariance restricts  $\chi_i$  to be functions of the scalars  $\pi^2, n^2$  and  $\pi \cdot p$ . There are, among several possibilities, two specially interesting gauges, namely

$$\chi_1^I = \pi \cdot p = 0, \chi_2^I = n^2 - a = 0,$$

$$\chi_1^{II} = \pi^2 - b = 0, \chi_2^{II} = n^2 - a = 0, \quad (7)$$

with  $a, b \in \Re$  and  $a < 0$ . In the case (I)  $S_\mu$  and  $p_\mu$  become parallel which makes the spin algebra anomalous:  $\{S_\mu, S_\nu\}_{DB}^I = 0$  and originates non-commuting coordinates ( $\{x_\mu, x_\nu\}_{DB}^I \neq 0$ ), mixing spin and space-time  $\{x_\mu, S_\nu\}_{DB}^I \neq 0$ . For the case (II) none of those problems appear. In particular, the canonical structure of  $(x_\mu, p_\nu)$  small phase space is maintained

and the  $\mathfrak{sl}(2, \mathfrak{R})$  spin algebra is not broken  $(\{S_\mu, S_\nu\}_{DB}^{II} = \epsilon_{\mu\nu\alpha} S^\alpha)$  which certainly simplifies the quantization. In both gauges the Poincaré covariance is not spoiled.

The above example teaches us that the spin algebra in a covariant theory for a free anyon is a gauge dependent quantity, and it is always possible to choose a gauge where the canonical  $\mathfrak{sl}(2, \mathfrak{R})$  spin algebra (3) is obeyed which is, as we have seen, a natural gauge choice avoiding complicate Dirac brackets. The  $\mathfrak{sl}(2, \mathfrak{R})$  spin algebra will guide us in the next section in finding a new rather simple action for a free anyon.

### 3 A Simple Action for an Anyon

According to the last section analysis of the degrees of freedom, if we extend the spinless particle phase space  $(x_\mu, p_\nu)$  introducing only one couple of conjugated variables  $\{q, \pi\} = 1$ , no extra constraints  $\Phi^A$  will be necessary besides the first class constraints  $\phi_1 = p^2 - m^2$  and  $\phi_2 = S \cdot p + \alpha m$  such that the Hamiltonian (4) becomes:

$$H(p^\mu, q, \pi) = \frac{e}{2} (p^2 - m^2) + \sigma (S_\mu(q, \pi) p^\mu + \alpha m). \quad (8)$$

In this minimally extended phase space we suppose that  $S_\mu(q, \pi)$  are analytical functions of  $q, \pi$  and satisfy the canonical  $\mathfrak{sl}(2, \mathfrak{R})$  spin algebra (3). The simplest Ansatz for  $S_\mu(q, \pi)$  compatible with those hypotheses is a linear function of  $\pi$ :

$$S_\mu = f_\mu(q)\pi + g_\mu(q), \quad (9)$$

with  $f_\mu(q)$  and  $g_\mu(q)$  analytic functions of  $q$  to be fixed by the algebra (3), that leads to the nonlinear differential equations:

$$f_\mu = \epsilon_{\mu\alpha\beta} f'^\alpha f^\beta, \quad (10)$$

$$g_\mu = \epsilon_{\mu\alpha\beta} g'^\alpha f^\beta, \quad (11)$$

where  $f'(q) = df/dq$ . After some algebra we show that the general solution to (11), compatible with (10) is

$$g_\mu(q) = r(q) f_\mu + s f'_\mu \quad (12)$$

where  $r(q)$  is an arbitrary function and  $s$  is an arbitrary real constant. Back in (9) we get  $S_\mu = (\pi + r(q))f_\mu + s f'_\mu$ . Since  $(q, \pi) \longrightarrow (q, \pi + r(q))$  is a canonical transformation that changes the Lagrangian by a total time derivative, we can set  $r(q) = 0$  without loss of generality from the physical point of view, such that in general

$$S_\mu = \pi f_\mu + s f'_\mu. \quad (13)$$

From (10)  $f_\mu$  must satisfy in particular:

$$f^2 = 0 ; f'^2 = -1 ; f'' \cdot f = 1. \quad (14)$$

Then we have

$$S^2 = -s^2. \quad (15)$$

Notice that the appearance of another arbitrary constant ( $s$ ) besides  $\alpha$  and  $m$  is not surprisingly since, due to the  $\mathfrak{sl}(2, \mathfrak{R})$  algebra (3), it is easy to deduce that the Casimir of the spin algebra  $S^2$  is an observable in the Poincaré multiplet, i. e.,  $\{S^2, J_\mu\} = 0 = \{S^2, P_\mu\}$ . In the last section covariant example the same arbitrary constant appear through the gauge conditions ( $\chi_i^{II} = 0$ ), resulting in :  $S^2 = a b$ .

Coming back to the differential equations (10), although we do not know their general solution, the simplest analytical solution (see identities (14)) is a second order polynomial <sup>2</sup> :

$$f_\mu^{(2)} = a_\mu q^2 + b_\mu q + c_\mu, \quad (16)$$

where for instance,  $a_\mu = (1, 0, -1)$ ;  $b_\mu = (-1, 1, 1)$ ;  $c_\mu = \frac{1}{2}(1, -1, 0)$ . It is worthwhile to point out that we do not really need to know the general solution for  $f_\mu(q)$  since any other realization  $\tilde{S}_\mu(\tilde{q}, \tilde{\pi})$ , with the same Casimir ( $\tilde{S}^2 = S^2$ ) will be physically equivalent to (16). Indeed, imposing  $\tilde{S}_\mu(\tilde{q}, \tilde{\pi}) = S_\mu^{(2)}(q, \pi) = \pi f_\mu^{(2)}(q) + s f_\mu'^{(2)}$ , two of these equations, for instance  $\tilde{S}_1 = S_1^{(2)}$  and  $\tilde{S}_2 = S_2^{(2)}$  will define two different relationships of

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<sup>2</sup> When quantized ( $\pi \sim d/dq$ ) these realizations correspond to the differential operators commonly used for the  $\mathfrak{so}(2,1)$  spectrum generating algebra, see for example [16].

the kind  $q^{(a)} = q^{(a)}(\tilde{q}, \tilde{\pi})$ ;  $\pi^{(a)} = \pi^{(a)}(\tilde{q}, \tilde{\pi})$  ( $a = 1, 2$ ). The identification  $\tilde{S}_\mu = S_\mu^{(2)}$  implies of course the identification of the corresponding Casimirs  $\tilde{S}^2 = (S^{(2)})^2$  which guarantees that  $\tilde{S}_0 = \pm S_0$ . The two signs correspond to the two different possibilities  $q^{(a)}, \pi^{(a)}$ . Therefore there will always be a canonical transformation bringing  $\tilde{S}_\mu(\tilde{q}, \tilde{\pi})$  to the form  $S_\mu^{(2)}(q, \pi)$ . Since this proof assumes no hypothesis whatsoever on the form of  $\tilde{S}_\mu(\tilde{q}, \tilde{\pi})$  it is clear now that any other more involved realization of  $\mathfrak{sl}(2, \mathfrak{R})$  algebra than (9) with  $\tilde{S}^2 = -s^2$  it is an unnecessary complication. Thus, we conclude that the Hamiltonian (8) is unique up to canonical transformations and it leads, by an inverse Legendre transformation, to a unique Lagrangian up to a total time derivative :

$$L = p^\mu \dot{x}_\mu + \pi \dot{q} - \frac{e}{2} (p^2 - m^2) - \sigma (\pi f \cdot p + s f' \cdot p + \alpha m). \quad (17)$$

The Legendre transformation is singular due to the Lagrangian constraint  $\sigma f \cdot \dot{x} = e \dot{q}$  but after a careful analysis, and eliminating some auxiliary fields we finally get:

$$L_s = \frac{1}{2e} \left( \dot{x}_\mu - \frac{s \dot{q} f'_\mu}{f \cdot \dot{x}} \right)^2 + \frac{e m^2}{2} - \frac{\alpha m e \dot{q}}{f \cdot \dot{x}}. \quad (18)$$

The quantity  $f_\mu(q)$  must be a solution of the equation (10), the Lagrangian  $L_s$  can be written in a polynomial form reintroducing the auxiliary fields. From  $L_s$  one derives two primary constraints  $\pi_e \approx 0$ ;  $\pi f \cdot p + s f' \cdot p + \alpha m = \phi_2 \approx 0$ , and the secondary one  $\phi_1 = p^2 - m^2 \approx 0$ ; as expected. The quantities  $S^2$ ,  $S \cdot p$  and  $p^2$  are dynamically independent from which it follows that  $L_s$  represents a continuous family of Lagrangians describing anyons with helicity  $\alpha$ , mass  $m$  and  $S_\mu S^\mu = -s^2$ . Specially simple is the case where  $S_\mu$  is light-like ( $s = 0$ )

$$L_0 = \frac{\dot{x}^2}{2e} + \frac{e m^2}{2} - \frac{\alpha m e \dot{q}}{f \cdot \dot{x}}. \quad (19)$$

It is remarkable that we just have to add the last term above with  $f_\mu(q)$  a solution of (10) like e. g. (16), in order to obtain an anyon from a relativistic spinless massive particle. The Lagrangians (18) and (19) are the main results of this work. The Lagrangian  $L_0$  is the simplest that we know describing a free anyon with the correct counting of degrees of freedom and that preserves



the  $\text{sl}(2, \mathfrak{R})$  spin algebra (3) and the canonical structure of the space-time variables (  $\{x_\mu, x_\nu\} = 0 = \{p_\mu, p_\nu\}$  ;  $\{x_\mu, p_\nu\} = \eta_{\mu\nu}$  ).

Due to the fact that  $f_\mu$  is a function of one variable  $f_\mu = f_\mu(q)$  which is a solution of equation (10), the Lagrangians (18) and (19), despite their appearance, are not Lorentz covariant. However, by performing a Lorentz rotation with constant parameter  $k^\mu$  ( $k^\mu = \epsilon^{\mu\nu\alpha} k_{\nu\alpha}$ ):

$$\delta x_\mu = \epsilon_{\mu\alpha\beta} k^\alpha x^\beta ; \delta q = k^\mu f_\mu(q). \quad (20)$$

it is easy to derive (using equation (10)):

$$\delta L_s = \frac{d}{dt} (-s f' \cdot k). \quad (21)$$

Therefore the action  $S = \int dt L_s$  is Lorentz, actually Poincaré, invariant. The Noether theorem will lead to the conserved quantities  $J_\mu$  and  $P_\mu$  of (2) with  $S_\mu$  given in (13). Those charges close the Poincaré algebra in terms of Poisson brackets. Thus, at least from the classical point of view, the Lagrangian  $L_s$  describes perfectly well defined relativistic invariant theories of free anyons, in a similar way to what happens to the Floreanini-Jackiw [17] action for chiral bosons or the Schwarz-Sen [18] action for electrodynamics. The loss of explicit Lorentz covariance was the price that we have paid for working in a small configuration space  $L_s(\dot{x}_\mu, q, \dot{q})$  instead of  $L(\dot{x}_\mu, n_\mu, \dot{n}_\alpha)$  of the covariant theory of previous section. In the next section we will return to the relation between our approach and the explicitly covariant one.

We finish this section analyzing the equations of motion of  $L_s$  which are equivalent to the Hamilton equations of

$$H_s = \frac{e}{2} (p^2 - m^2) + \sigma (S_\mu(q, \pi) p^\mu + \alpha m) + \mu \pi_e, \quad (22)$$

where  $S_\mu(q, \pi)$  is given in (13). It is easy to derive that

$$\pi = -\frac{(s f' \cdot p + \alpha m)}{f \cdot p} ; \dot{q}(t) = \sigma f^\mu(q) p_\mu \quad (23)$$

$$\dot{x}_\mu = e p_\mu + \sigma S_\mu ; \dot{p}_\mu = 0, \quad (24)$$

besides, the constraints  $p^2 - m^2 = 0$  and  $\pi_e = 0$ . Even in the simplest case when  $f_\mu(q) = f_\mu^{(2)}(q)$  the equation for  $q(t)$  is a bit complicated, but writing it in terms of  $S_\mu$  we have:

$$\dot{S}_\alpha = \sigma \epsilon_{\alpha\mu\gamma} p^\mu S^\gamma, \quad (25)$$

consequently:

$$\ddot{S}_\alpha - \frac{\dot{\sigma}}{\sigma} \dot{S}_\alpha + m^2 \sigma^2 S_\alpha = -\alpha m \sigma^2 p_\alpha. \quad (26)$$

In a gauge where  $\dot{\sigma} = 0 = \dot{e}$  we have the solution:

$$S_\alpha(t) = -\frac{\alpha}{m} p_\alpha + \frac{\sigma}{\omega} \left( p^2 c_\alpha - p \cdot c p_\alpha \right) \cos(\omega t) + \epsilon_{\alpha\beta\gamma} p^\beta c^\gamma \sin(\omega t) \quad (27)$$

$$x_\mu(t) = x_\mu(0) + \left( e - \frac{\alpha}{m} \sigma \right) p_\mu t + \left( \frac{\sigma}{\omega} \right)^2 \left( p^2 c_\alpha - p \cdot c p_\alpha \right) \sin(\omega t) + \\ - \frac{\sigma}{\omega} \left( \epsilon_{\alpha\beta\gamma} p^\beta c^\gamma \right) \cos(\omega t), \quad (28)$$

where  $\omega = m \mid \sigma \mid$  and  $c_\alpha$  is a constant vector with only one independent component which is related to  $q(0)$ . Therefore, in agreement with [11], in order to interpret  $x_\mu(t)$  as center of mass coordinates of a free rigid body ( $\ddot{x}_\mu = 0$ )  $S_\alpha$  and  $p_\alpha$  must be parallel but in general this is not the case and the oscillating motion (Zitterbewegung) will take place along the orthogonal directions to  $p_\mu$ .

## 4 Connection to other Lagrangians

The simplicity of  $L_o$  when compared to other approaches like the covariant one of [8] rises up the question about the link between these theories. It will be seen that there is a rather interesting and direct connection between these formulations. In order to compare the covariant approach sketched in section 2 to  $L_o$  we have to fix the gauge of the constraints  $\pi \cdot n = 0$  and  $p \cdot n = 0$  (see (6)) such that the  $SL(2, \mathfrak{R})$  spin algebra is preserved and furthermore  $S^2 = (\epsilon_{\mu\nu\alpha} n^\alpha \pi^\alpha)^2 = n^2 \pi^2 - (\pi \cdot n)^2 = 0$ . The gauge conditions  $\chi_1 = \pi^2 = 0$ ;  $\chi_2 = n^2 - a = 0$  ( $a$  is a negative constant) will enforce the two previous requirements. We end up with four second class constraints:

$$n^2 - a = 0 \quad ; \quad \pi^2 = 0 \quad (29)$$

$$p \cdot n = 0 \quad ; \quad \pi \cdot n = 0 \quad (30)$$

Those can be explicitly solved for, e.g. ,  $n_\mu$  and  $\pi_0$  as functions of  $\pi_1$  and  $\pi_2$ :

$$n_\mu = \pm \sqrt{-a} \frac{\epsilon_{\mu\nu\alpha} p^\nu \pi^\alpha}{\pi \cdot p} \quad ; \quad \pi_0 = \pm \sqrt{\pi_1^2 + \pi_2^2} \quad (31)$$

which imply  $S_\alpha = \pm\sqrt{-a}\pi_\alpha$ . Henceforth  $n_\mu$  and  $\pi_0$  will be understood as (31). The  $sl(2, \mathbb{R})$  algebra for  $S_\mu$  follows from the Dirac brackets:

$$\{\pi_\alpha, \pi_\beta\}_{DB} = -\frac{\epsilon_{\alpha\beta\gamma}S^\gamma}{a} = \pm\frac{\epsilon_{\alpha\beta\gamma}\pi^\gamma}{\sqrt{-a}} \quad (32)$$

Inserting the solutions (31) into the first order form of the covariant Lagrangian of section 2, namely,

$$L = p \cdot \dot{x} + \pi \cdot \dot{n} - \frac{e}{2}(p^2 - m^2) - \sigma(S \cdot p + \alpha m) - \lambda_1 \pi \cdot n - \lambda_2 p \cdot n \quad (33)$$

we derive, after elimination of  $p_\mu$ ,

$$\tilde{L} = \frac{(\dot{x}_\mu - \sqrt{-a}\sigma\pi_\mu)^2}{2e} + e m^2 - \alpha m \sigma + \frac{\sqrt{-a}}{\pi \cdot \dot{x}} \epsilon_{\alpha\beta\gamma} \dot{x}^\alpha \dot{\pi}^\beta \pi^\gamma \quad (34)$$

It can be shown that this intermediate Lagrangian corresponds exactly to the one written in formula (4.2) of [12] for the case of lightlike spin ( $S^\mu S_\mu = 0$ ) where the following assignments are understood:  $\sigma = v$ ;  $\sqrt{-a}\pi_\mu = j_\mu$ . We emphasize that  $\tilde{L}$  is not explicitly Lorentz covariant despite its appearance since  $\pi_0 = \pm\sqrt{\pi_1^2 + \pi_2^2}$  and that makes it more complicated than it looks like. Now a comment is in order; The hamiltonian reduction from  $(n_\mu, \pi_\nu)$  to  $(\pi_1, \pi_2)$  keeps the constraint  $\phi_2 = S \cdot p + \alpha m \approx 0$  first class, thus it should furnish a similar Lagrangian to  $L_0$ . The fact that  $\tilde{L}$  depends on two extra dynamical variables besides  $x_\mu$  ( $L = L(x_\mu, \pi_1, \pi_2)$ ), instead of just one like  $L_0 = L_0(x_\mu, q)$  is a consequence of the Dirac brackets (32) which show that  $\pi_1$  and  $\pi_2$  are not canonically conjugated variables like the couple  $(q, \pi)$  of section 3. That is one of the sources of complications in the intermediate Lagrangian  $\tilde{L}$ . Indeed, comparing the definition of spin of our approach for  $s = 0$  with the case of  $\tilde{L}$  ( $S_\mu = \sqrt{-a}\pi_\mu$ ) we have the identification:

$$\pi_\mu = \frac{\pi f_\mu(q)}{\sqrt{-a}} \quad (35)$$

which when introduced in  $\tilde{L}$  reduces it to precisely  $L_0$ . Therefore we conclude that  $L_0$  can be obtained directly from the covariant theory reviewed in section 2 through a gauge fixing procedure followed by the non canonical transformation (Darboux transformation) (35) that brings the non canonical

couple  $(\pi_1, \pi_2)$  into the canonical one  $(q, \pi)$ . Dynamically the situation is that on one hand, after the gauge fixing, both  $\pi_1$  and  $\pi_2$  propagate while after the Darboux transformation (35) on the other hand only  $q$  propagates and  $\pi$  can be eliminated by its equation of motion. The extra constraints which certainly appear in the first case are responsible for the match of the degrees of freedom in both cases.

Last we comment upon the case  $(S^\mu S_\mu = -s^2 \neq 0)$ . Reintroducing auxiliary fields,  $L_s$  can be rewritten as:

$$L_s = \frac{(\dot{x}_\mu - \sigma S_\mu)^2}{2e} + \frac{em^2}{2} - \alpha m \sigma + \pi \dot{q} \quad (36)$$

where  $S_\mu = \pi f_\mu + s f'_\mu$ . Comparing with the intermediate Lagrangian of [12] we are led to another Darboux transformation similar to the previous one,

$$j_\mu = \pi f_\mu(q) + s f'_\mu(q) \quad (37)$$

where  $j_\mu$  are the variables used in [12] for the spin and they are such that  $j^2 = \text{constant}$ . Introducing (37) in the Lagrangian of [12] it is easy to show that:

$$L_s = L([12]) - \frac{d}{dt} \left[ \frac{s}{2} \ln \frac{f \cdot \xi}{(S \cdot \xi - s^2)} \right] \quad (38)$$

where  $\xi_\mu$  is a constant vector introduced in [12] which satisfies  $\xi^2 = -1$ . Therefore the two Lagrangians are dynamically equivalent, though  $L_s$  is much simpler. Now two remarks should be made: First, although we believe that it is possible, we have not been able to generalize the gauge conditions (29) to the  $s \neq 0$  case and consequently we do not know how to get the intermediate Lagrangian from the covariant one as we did for the  $s = 0$  case. Secondly, the canonical pair  $(q, \pi)$  is similar to the canonical variables  $(J^{(0)}, \varphi)$  of [12] nevertheless there is no canonical transformation between these two couples since  $(J^{(0)}, \varphi)$  were obtained from the two independent spin variables by means of a transformation which includes the momentum  $p_\mu$  differently from our simpler variables.

## 5 Summary and Outlook

By minimally extending the space-time phase space  $(x_\mu, p_\nu)$  of a spinless massive particle through the introduction of one couple of canonical variables

$(q, \pi)$  we have derived a continuous family of Lagrangians  $L_s = L_s(\dot{x}_\mu, q, \dot{q})$  that describe a free anyon with given mass  $m$  and given helicity  $\alpha$ . The Lagrangian  $L_s$  has a number of nice features like : correct counting of degrees of freedom ; canonical spin algebra ( $\{S_\alpha, S_\beta\} = \epsilon_{\alpha\beta\gamma} S^\gamma$ ) ; canonical space time structure ( $\{x_\alpha, x_\beta\} = 0 = \{p_\alpha, p_\beta\}$ ;  $\{x_\alpha, p_\beta\} = \eta_{\alpha\beta}$ ) and specially remarkable is the simplicity of the case where the dual spin vector is light like ( $S^2 = -s^2 = 0$ ). The price we have paid for all those nice features was the loss of explicit Lorentz covariance, though we have shown that the action obtained is indeed Lorentz invariant very much like the Floreanini and Jackiw [17] action for chiral bosons and the Schwarz and Sen action [18] for electrodynamics.

A key ingredient to obtain  $L_s$  was the study of realizations of the  $SL(2, \mathbb{R})$  algebra in a two dimensional phase space like  $(q, \pi)$ . It was shown that, up to a canonical transformation, the realizations which have the same Casimir can be taken to be of the form (13) with  $f_\mu$  given in (10). Therefore, given  $s, \alpha$ , and  $m$  the action  $S_s = \int L_s dt$  is physically equivalent, at the classical level, for any realization  $\tilde{S}_\mu(\tilde{q}, \tilde{\pi})$  of the  $SL(2, \mathbb{R})$  algebra such that the Casimir is the same, i.e.,  $\tilde{S}^2 = S^2 = -s^2$ .

In section 4 we have shown that the Lagrangian  $L_0$  can be alternatively obtained from the more involved covariant description of anyons of [12] by means of appropriate gauge conditions followed by a Darboux transformation. The generalization to the case where  $S^2 = -s^2 \neq 0$  is still under investigation as well as the search for Lagrangians similar to  $L_s$  that describe free anyons with time like spin vectors ( $S^2 > 0$ ).

Finally we mention that the natural sequence for this work is the quantization of  $L_s$ . Especially promising is the calculation of the anyon propagator that although non-covariant may give some clue on the covariant case and the corresponding field theoretical action for the relativistic free anyon. Of course, it is also of interest to couple  $L_s$  to an external electromagnetic field whose non-relativistic limit may be important for practical applications of anyons.

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